# Dynamo mechanism: Effects of correlations and viscosities

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We analyze the effects of the background velocity and the initial magnetic field correlations, and viscosities on the turbulent dynamo and the $\alpha$-effect. We calculate the $\alpha$-coefficients for arbitrary magnetic and fluid viscosities, background velocity and the initial magnetic field correlations. We explicitly demonstrate that the general features of the initial growth and late-time saturation of the magnetic fields due to the non-linear feedback are qualitatively independent of these correlations. We also examine the hydrodynamic limit of the magnetic field growth in a renormalization group framework and discuss the possibilities of suppression of the dynamo growth below a critical rotation. We demonstrate that for Kolmogorov- (K41) type of spectra the *Ekman number* $M \overset{>}{\sim} 1/2$ for dynamo growth to occur.

## I. INTRODUCTION

Magnetic fields are ubiquitous. All astrophysical objects are known to have magnetic fields of different magnitudes,e.g., 1 gauss at the stellar scale to $10^{-6}$ gauss at the galactic scale [1]. The origin of such fields (*primordial field*) is not very clear - there are several competing theories which attempt to describe this [2]. However, a finite magnetic field in any physical system undergoes a temporal decay due to the finite conductivity of the medium. So, for steady magnetic fields to occur in astrophysical bodies, there has to be a mechanism of regeneration of the magnetic fields, which takes place due to the dynamo process [1,3]. Most astrophysical bodies are thought to have *fast* dynamo operating within themselves (there are exception to this, e.g., the Moon, Venus and Mars in our solar system) resulting into exponential growth of the magnetic fields. This mechanism requires a turbulent velocity background [1] [though non-turbulent velocity fields too can make a seed (initial) magnetic field to grow (for details see [3]), we will not consider such cases here]. Since the dynamo equation, in the linear approximation (see below) gives unbounded exponentially growing solutions for the long wavelength (large scale) part of the magnetic fields, it is linearly unstable in the low wavenumber limit. However, one does not see a perpetual growth of magnetic fields in the core of the earth or in the sun. For example, geomagnetic fields ($\sim$ 1 gauss) are known to be stable for about $10^6$ years [1]. Thus, the physically realisable solutions of the dynamo equations cannot be unstable in the long time limit. It is now believed that the non-linear feedback due the Lorentz force term in the Navier-Stokes equation is responsible for the saturation of the magnetic field growth (see, e.g., [1]).

The study of this problem has already been the subject of previous work by many groups. For example Pouquet, Frisch and Léorat [4] studied the connections between the dynamo process and the inverse cascade of magnetic and kinetic energies within a eddy damped quasi-normal Markovian approximation. Moffatt [5] has examined the back reactions due to the Lorentz force for magnetic Prandtl number $P_m \gg 1$ by linearising the equations of motion of three-dimensional ($3d$) magnetohydrodynamics (MHD). Vainshtein and Cattaneo [6] discussed several nonlinear restrictions on the generations of magnetic fields. Field *et al* [7] discussed nonlinear $\alpha$-effects within a two-scale approach. Rogachevskii and Kleeorin [8] studied the effects of an anisotropic background turbulence on the dynamo process. Brandenburg examined non-linear $\alpha$-effects in numerical simulation of helical MHD turbulence [9]. In particular, he examined the dependences of dynamo growth and the saturation field on the magnetic Prandtl number $P_m$ (the ratio of the magnetic- to the kinetic- viscosities). Bhattacharjee and Yuan [10] studied the problem in a two-scale approach by linearising the equations of motion.

Dynamo mechanism has two competing processes at work: amplification of the magnetic field by the dynamo process and ohmic dissipation due to finite resistivity of the medium concerned. Which one among these two effects will dominate depends on the case in study. In some specific models, however, one can analyze this completely. A good example of such models is the Kraichnan-Kazantzev dynamo [11,12] where the velicity field is assumed to be Gaussian-distributed, delta-correlated in time and the magnetic field is governed by the Induction equation [22]. In this model the statistics of the velocity field is taken to be parity invariant so that the $\alpha$-effect is ruled out. The main results from this model include i)the existence of dynamo in the infinite magnetic Reynolds number limit for a

particular choice of the variance of the velocity distribution [13] and ii)the existence of a critical magnetic Reynolds number only above which dynamo growth is possible [14]. However, not much is known about this when invariance due to parity is broken and when the velocity field is not temporally delta-correlated. In a recent simulations [15] the authors found, in a model simulation for the solar convection zone, a monotonic increase of the horizontal $\alpha$-effect with rotation. Kida *et al* showed, in numerical simulations, that unless magnetic hyperviscosity is less than a critical value, magnetic fields did not grow [31], confirming the existence of a critical magnetic Reynolds number ($R_m$).

Our studies generalize the existing results. In this paper we use a minimal model of $\alpha$-effect (see below) to study dynamo with $\alpha$-effect to calculate the $\alpha$ coefficient for arbitrary correlations and viscosities, and ask the following questions:

1. Do the turbulent dynamo growth and the saturation processes require any turbulent background? Or do they function with arbitrary parity-breaking and fluctuating velocity and initial magnetic field correlations? [1] .

2. What is the *hydrodynamic limit* (long wavelength limit) of the dynamo problem? By this we ask how the magnetic field correlations scale in the infra red limit during the initial-growth regime.

3. Can arbitrarily large magnetic viscosity prevent dynamo growth? In other words, is there a critical magnetic Reynolds number $R_m$ above which the dynamo growth sets in?

To study the above mentioned questions we employ a diagrammatic perturbation theory, which has been highly successful in the contexts of critical dynamics [18], driven systems [19], etc. This can be easily extended to higher orders in perturbation expansion and is very suitable for handling continuous kinetic and magnetic spectra, unlike the two-scale approximation. This was first used to study stationary, homogeneous and isotropic MHD in Ref. [20]. We use this method to study non-stationary statistical states (dynamo growth) which facilitates studies on the hydrodynamic limit of the dynamo problem in a renormalisation group framework. We use diagrammatic perturbation theory to calculate expressions for the $\alpha$ coefficiants for arbitrary background velocity and initial magnetic field correlations and magnetic Prandtl number $P_m$ for both the early growth and the late time saturation. With our expressions for $\alpha$ we examine the three issues mentioned above.

We investigate these for arbitrary correlations and magnetic Prandtl number $P_m$ with no approximations other than the existence a perturbation theory. Our principal results are:

- We calculate the $\alpha$-coefficients for arbitrary correlations and viscosities.

- We examine the hydrodynamic limit in the kinematic regime and predict the existence of a critical $R_m$ or rotation above which dynamo growth will occur for certain correlations with infra red singularity.

In our all our studies, we do not assume any variance for the velocity field. Instead, we use the Navier-Stokes equation to describe the dynamics of the velocity field. This allows us to use a renormalisation group framework to study the hydrodynamic limit.

The first question that we investigate is phenomenologically very important because different systems may have different velocity and initial magnetic field spectra. Therefore, it is important to understand the dependence of the dynamo on these spectra. We explicitly demonstrate that the nonlinear feedback of the magnetic fields on the velocity fields in the form of the Lorentz force stabilises the growth for arbitrary velocity and initial magnetic field correlations. This demonstrates that the basic features of the dynamo mechanism are qualitatively independent of the velocity and magnetic field spectra and, essentially, are properties of the $3d$MHD equations. Details (e.g., the values of the $\alpha$-coefficients) of course, depend upon the actual forms of the spectra. Our renormalization group analysis indicates that dynamo growth takes place only if the Ekman number $M \lesssim 1/2$ (for a given $R_m$) when the velocity and the initial magnetic field spectra are sufficiently singular in the long wavelength limit. The structure of this paper is as follows: In Sec.II we discuss the general dynamo mechanism within the standard linear approximation for arbitrary velocity and initial magnetic field correlations and viscosities. In Sec.III B we show that beyond the linear approximation non-linear effects lead to the eventual saturation of magnetic field growth for arbitrary background kinetic energy and initial magnetic energy spectra, and viscosities. We elucidate how different background kinetic energy and initial

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[1]By a turbulent background we do not mean any kind of fluctuating state but a fluctuating state with Kolmogorov (K41) spectra $\propto k^{-5/3}$ for the kinetic and magnetic energies and cascades of appropriate quantities; if there is no mean magnetic field then the energy spectra is expected to be K41-type - see Ref. [16].

magnetic energy spectra affect the values of the $\alpha$-coefficients. In Sec.IV we analyze the initial dynamo growth in a renormalization group framework. We show that for sufficiently singular velocity and magnetic field spectra the Ekman number must be $\lesssim 1/2$ for the magnetic fields to grow. For velocity and magnetic field spectra which go to zero in the long wavelength limit there are no such restrictions. In Sec.V we present our conclusions.

## II. DYNAMO GROWTH: THE LINEAR APPROXIMATION

In the kinematic approximation [1,21], i.e., in the early-time regime, when the magnetic energy is much smaller than the kinetic energy ($\int u^2 d^3r >> \int b^2 d^3r$, where $\mathbf{u}(\mathbf{r},t)$ and $\mathbf{b}(\mathbf{r},t)$ are the velocity and magnetic fields respectively) the Lorentz force term of the Navier Stokes equation is neglected. In that weak magnetic field limit, which is reasonable at an early time, the time evolution problem for the magnetic fields is a linear problem as the Induction equation [22] is linear in magnetic fields $\mathbf{b}$:

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{b}) + \mu \nabla^2 \mathbf{b}, \tag{1}$$

where $\mu$ is the magnetic viscosity. The velocity field is governed by the Navier-Stokes equation [23] (in the absence of the Lorentz force)

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla \mathbf{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u} + \mathbf{f}. \tag{2}$$

Here $\nu$ is the fluid viscosity, $\mathbf{f}$ an external forcing function, $p$ the pressure and $\rho$ the density of the fluid. We take $\mathbf{f}$ to be a zero mean, Gaussian stochastic force with a specified variance (see below).

In a two-scale [1] approach one can then write an *effective* equation for $\mathbf{B}$, the long-wavelength part of the magnetic fields [1]:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \nabla \times \mathbf{E} + \mu \nabla^2 \mathbf{B}, \tag{3}$$

where the *Electromotive force* $\mathbf{E} = \langle \mathbf{u} \times \mathbf{b} \rangle$. $\mathbf{U}$ is the large scale component of the velocity field $\mathbf{u}$. An *Operator Product Expansion* (OPE) is shown to hold [21] which provides a gradient expansion in terms of $\mathbf{B}$ for the product $\mathbf{E} = \langle \mathbf{u} \times \mathbf{b} \rangle$ [1]

$$E_i = \alpha_{ij} B_j + \beta_{ijk} \frac{\partial B_j}{\partial x_k} + .... \tag{4}$$

For homogenous and isotropic flows ($\alpha_{ij} = \alpha \delta_{ij}$) Eq.(4) gives,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \alpha \nabla \times \mathbf{B} + \mu \nabla^2 \mathbf{B}, \tag{5}$$

which is the standard turbulent dynamo equation. Here $\mu$ now is the *effective* magnetic viscosity which includes both the microscopic magnetic viscosity and the turbulent diffusion, represented by $\beta_{ijk}$ in Eq.(4). $\alpha$ depends upon the statistics of the velocity field (or, equivalently, the correlations of $\mathbf{f}$). Retaining only the $\alpha$-term and dropping all others from the RHS of Eq.(5), the equations for the cartesian components of $\mathbf{B}$ become (we neglect the dissipative terms proportional to $k^2$ as we are interested only in the long wavelength properties)

$$\frac{d}{dt}\begin{pmatrix} B_x(\mathbf{k},t) \\ B_y(\mathbf{k},t) \\ B_z(\mathbf{k},t) \end{pmatrix} = i\alpha \begin{pmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{pmatrix} \begin{pmatrix} B_x(\mathbf{k},t) \\ B_y(\mathbf{k},t) \\ B_z(\mathbf{k},t) \end{pmatrix}.$$

The eigenvalues of the matrix is $\lambda = \pm ik$, 0. Thus depending on the sign of the product $\alpha k$, one mode grows and the other decays. The third mode is unphysical, because the corresponding eigenfunction is proportional to $\mathbf{k}$ and hence in conflict with $\nabla \cdot \mathbf{B} = \mathbf{0}$. Since growth rate is proportional to $|k|$ and dissipation is proportional to $k^2$, large scale fields continue to grow leading to long wavelength instability. Thus in the long time limit effectively only the growing mode remains. Growth rate $\alpha$ is a pseudo-scalar quantity, i.e., under parity transformation $\mathbf{r} \to -\mathbf{r}$, $\alpha \to -\alpha$ [1,21]. Since $\alpha$ depends upon the statistical properties of the velocity field, its statistics should not be parity invariant. This can happen in a rotating frame, where the angular velocity explicitly breaks reflection invariance.

## III. FORMULATION OF THE DYNAMO PROBLEM IN A ROTATING FRAME

The Navier-Stokes (NS) (including the Lorentz force) and the Induction equation in an inertial frame in $(\mathbf{k}, t)$ space take the form

$$\frac{\partial u_i(\mathbf{k}, t)}{\partial t} + \frac{1}{2}P_{ijp}(\mathbf{k})\sum_q u_j(\mathbf{q}, t)u_p(\mathbf{k} - \mathbf{q}, t) = \frac{1}{2}P_{ijp}(\mathbf{k})\sum_q b_j(\mathbf{q}, t)b_p(\mathbf{k} - \mathbf{q}, t) - \nu k^2 u_i + f_i(\mathbf{k}, t), \tag{6}$$

and

$$\frac{\partial b_i(\mathbf{k}, t)}{\partial t} = \tilde{P}_{ijp}(\mathbf{k})\sum_q u_j(\mathbf{q}, t)b_p(\mathbf{k} - \mathbf{q}, t) - \mu k^2 b_i. \tag{7}$$

Here, $u_i(\mathbf{k}, t)$ and $b_i(\mathbf{k}, t)$ are the fourier transforms of $u_i(\mathbf{r}, t)$ and $b_i(\mathbf{r}, t)$ respectively, $P_{ijp}(\mathbf{k}) = P_{ij}(\mathbf{k})k_p + P_{ip}(\mathbf{k})k_j$, $\tilde{P}_{ijp}(\mathbf{k}) = P_{ij}(\mathbf{k})k_p - P_{ip}(\mathbf{k})k_j$, $P_{ij}$ is the projection operator, which appears due to the divergence-free conditions on the velocity and magnetic fields (we consider incompressible fluid for simplicity). Equations (6) and (7) have to be supplemented by appropriate correlations of $f_i$ and initial conditions on $b_i$. We choose $f_i(\mathbf{k}, t)$ and $b_i(\mathbf{k}, t = 0)$ to have zero mean and to be Gaussian distributed with the following variances:

$$\langle f_i(\mathbf{k}, t)f_j(-\mathbf{k}, 0)\rangle = 2P_{ij}D_1(k)\delta(t), \tag{8}$$

$$\langle b_i(\mathbf{k}, t = 0)b_j(-\mathbf{k}, t = 0)\rangle = 2P_{ij}D_2(k), \tag{9}$$

where $D_1$ and $D_2$ are some functions of $k$ (to be specified later).

In a rotating frame with a rotation velocity $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$ the Eq.(6) takes the form

$$\frac{\partial u_i(\mathbf{k}, t)}{\partial t} + 2(\boldsymbol{\Omega} \times \mathbf{u})_i + \frac{1}{2}P_{ijp}(\mathbf{k})\sum_q u_j(\mathbf{q}, t)u_p(\mathbf{k} - \mathbf{q}, t) = \frac{1}{2}P_{ijp}(\mathbf{k})\sum_q b_j(\mathbf{q}, t)b_p(\mathbf{k} - \mathbf{q}, t) + \nu\nabla^2 u_i + f_i(\mathbf{k}, t), \tag{10}$$

whereas Eq.(7) has the same form in the rotating frame. $\boldsymbol{\Omega} \times \mathbf{u}$ is the coriolis force. The centrifugal force $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$ is a part of the *effective pressure*$= p + \frac{1}{2}|\boldsymbol{\Omega} \times \mathbf{r}|^2$ which does not contribute to the dynamics of incompressible flows. The bare propagator $G_u$ (obtained from the linearized version of Eq.(10)) of $u_i$

$$G_u = \begin{pmatrix} \frac{i\omega + \nu k^2}{(i\omega + \nu k^2)^2 + 4\Omega^2} & -\frac{2\Omega}{(i\omega + \nu k^2)^2 + 4\Omega^2} & 0 \\ \frac{2\Omega}{(i\omega + \nu k^2)^2 + 4\Omega^2} & \frac{i\omega + \nu k^2}{(i\omega + \nu k^2)^2 + 4\Omega^2} & 0 \\ 0 & 0 & \frac{1}{i\omega + \nu k^2} \end{pmatrix}$$

such that $\mathbf{u} = \mathbf{G_u}\mathbf{f}$ where $\mathbf{u}$ is the column vector

$$\mathbf{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}.$$

One can verify that with the form of the bare propagator given above, an odd-parity part in the velocity auto-correlator $\langle u_i(\mathbf{k}, t)u_j(-\mathbf{k}, 0)\rangle$ appears which is proprotional to the rotation $\Omega$. Notice that $G_u^{zz}$ is different from $G_u^{xx,yy}$ - this is just the consequence of the fact that $\Omega$ distinguishes the $z$-direction as a preferred direction in space, making the system anisotropic. However for frequencies $\omega >> \Omega$ or length scales $k^z >> \Omega$ (here $z$ is the dynamical exponent) isotropy is restored. In that regime, to $O(\Omega)$ the role of the global rotation is to introduce a non-zero odd-parity part in $\langle u_i u_j \rangle$ proportional to $\Omega$. This can be also seen by noting that in the inertial frame the correlation $\langle u_i(\mathbf{k})u_j(-\mathbf{k})\rangle$ is of the form $P_{ij}(\mathbf{k})A(k)$ [cf. Eq.(8)] where $A(k)$ is a scalar function of $k$ and hence in the rotating frame the correlator is proportional to $RP_{ij}R^T$ where $R$'s are appropriate rotation matrices (we have suppressed the indices). Similarly, initial magnetic field correlations, given by Eq.(9) transforms accordingly in the rotating frame. Since rotation matrices act on $\langle u_i(\mathbf{k})u_j(-\mathbf{k})\rangle$ and Eq.(9) in the same way, magnetic field auto-correlator $\langle b_i(\mathbf{k}, t)b_j(-\mathbf{k}, 0)\rangle$ has an odd parity part in the rotating frame with the same sign as the odd parity part in the velocity correlator.

Thus the effects of rotation can be *modeled* (to the lowest order) by introducing parity breaking parts in Eqs.(8) and (9) [1]

$$\langle f_i(\mathbf{k}, t) f_j(-\mathbf{k}, 0) \rangle = 2 P_{ij} D_1(k) \delta(t) + 2i\epsilon_{ijp} k_p \tilde{D}_1(k) \delta(t),$$
$$\langle b_i(\mathbf{k}, t=0) b_j(-\mathbf{k}, t=0) \rangle = 2 P_{ij} D_2(k) \delta(t) + 2i\epsilon_{ijp} k_p \tilde{D}_2(k), \tag{11}$$

in conjunction with the Eqs.(6) and (7), where $\epsilon_{ijp}$ is the totally antisymmetric tensor in $3d$. This way of modeling rotation effects is, of course, only approximate, but suffices for our purposes as this explicitly incorporates parity breaking. One can, however, construct experimental set ups [1] which are described correctly by Eqs.(11). The parity breaking parts in the noise correlations or initial conditions ensure that the velocity and the initial magnetic field correlators have non-zero odd parity parts, as would happen in a rotating frame. An important dimensionless number is the *Ekman number* $M = \frac{\nu L^2}{2\Omega}$ which can be related to $\tilde{D}_1$ by equating the parity braking parts of the velocity correlator calculated from (linearized) Eq.(10) and Eq.(8) with that from Eqs. (6) and (11). This gives $\tilde{D}_1 = 2M^{-1} D_1$. Now, one may ask what is the relative sign between $\tilde{D}_1$ and $\tilde{D}_2$? Since the parity breaking parts of the correlators of the velocity and the magnetic fields have same sign and are proportional to $\tilde{D}_1$ and $\tilde{D}_2$ respectively, $\tilde{D}_1$ and $\tilde{D}_2$ must have same sign. As already noted, introduction of parity breaking terms in the force/initial correlations is well-known in the literature, we, nevertheless, give the analysis in details in order to emphasise on the fact that fluid and magnetic helicities must have the same sign. Furthermore, for a complete description of the effects of rotation, in addition to the coriolis force, a forcing with a preferred direction is also required. We, however, do not include all these details as introduction of parity-breaking correlations is sufficient for our purposes. In this sense, this can be thought of as a *reduced* or a *minimal model* for dynamo. One may note that a nonzero kinetic helicity is required for the $\alpha$-effect as the $\alpha$-coefficient is proportional to the kinetic helocity. Even though a global rotation explicity breaks the parity invariance of the system under space reversal, rotation alone is not enough to yield a non-zero helicity. This is because the helicity is pseudo-scalar and, therefore, can be constructed only out of an axial vector (here, rotation $\mathbf{\Omega}$) and a polar vector. In typical astrophysical settings, the latter one could be provided by, say, a density inhomogeneity. Even though this is not contained in Eq. (6), our minimal model, nevertheless, produces a finite helicity due to the helical nature of the forcing function. Thus, our minimal model is able to capture both the breakdown of parity due to the rotation and the generation of helicity due to the rotation and any other preferred direction.

## A. The $\alpha$ in the kinematic approximation: Dependences on background velocity and initial magnetic field spectra

In the kinematic approximation, which neglects the Lorentz force term of the Navier-Stokes equation, the time evolution of the magnetic fields follows from the linear Induction Equation (1). We assume, for the convenience of calculations, that the velocity field ($\mathbf{u}$) has reached a statistical steady state. This is acceptable as long as the loss due to the transfer of kinetic energy to the magnetic modes by the dynamo process is compensated by the external drive. In the kinematic (i.e., linear) approximation, we work with the Eqs.(6) (without the Lorentz force) and (7). We choose $f_l(\mathbf{k}, t)$ to be a zero-mean, Gaussian random field with correlations

$$\langle f_l(\mathbf{k}, t) f_m(\mathbf{k}, 0) \rangle = 2 P_{lm} D_1(k) \delta(t) + 2i\epsilon_{lmn} k_n \tilde{D}_1(k) \delta(t). \tag{12}$$

Our initial conditions for the magnetic fields are

$$\langle b_\alpha(\mathbf{k}, t=0) b_\beta(-\mathbf{k}, t=0) \rangle = 2 P_{\alpha\beta} D_2(k) + 2i\epsilon_{\alpha\beta\gamma} k_\gamma \tilde{D}_2(k), \tag{13}$$

Since we are interested to investigate the dynamo process with arbitrary statistics for the velocity and magnetic fields we work with arbitrary $D_1(k), \tilde{D}_1(k), D_2(k)$ and $\tilde{D}_2(k)$. For K41-type spectra, we require [24] $D_1(k) = D_1 k^{-3}, \tilde{D}_1(k) = \tilde{D}_1 k^{-4}, D_2(k) = D_2 k^{-5/3}$ and $\tilde{D}_2(k) = k^{-8/3}$. These choices ensure that under spatial rescaling $\mathbf{x} \to l\mathbf{x}$, $\mathbf{u}, \mathbf{b} \to l^{1/3} \{\mathbf{u}, \mathbf{b}\}$ which is the Kolmogorov scaling [24]. Note that both the force correlations in the Eq.(6) and the initial conditions on Eq.(7) have parts that are parity breaking, in conformity with our previous discussions. We now calculate the $\alpha$-term. We use an iterative perturbative method which is very similar to and discussed in details in Ref. [19]. In this method, terms in each order of the perturbation series can be represented by appropriate Feynman diagrams [19]. Even though, for simplicity, we confine ourselves to the lowest order in the perturbation theory (represented by the *tree level diagrams*), which is sufficient for our purposes, higher order calculations represented by higher order digrams can be done in a straight forward manner. Below we give the expression for $\alpha$

in the kinematic approximation (which we call the 'direct' term - responsible for growth) in the lowest order of the perturbation theory (see Fig.1a):

$$\langle (\mathbf{u} \times \mathbf{b})_\mu \rangle_D = \langle \int_{q,q_1} \epsilon_{\mu\beta\gamma} u_\beta(\mathbf{q},t) b_\gamma(\mathbf{k}-\mathbf{q},t) \rangle = \langle \int_q \epsilon_{\alpha\beta\gamma} u_\beta(\mathbf{q},t) \epsilon_{\gamma\delta\lambda} i(\mathbf{k}-\mathbf{q})_\delta u_\eta(\mathbf{q_1},t_1) b_\tau(\mathbf{k}-\mathbf{q}-\mathbf{q_1},t_1) G_0^b(k-q,t-t_1) \rangle$$

(14)

from which one can read the $\alpha$-term:

$$\alpha_D B_\alpha(\mathbf{k},t) = \int_q \frac{i\tilde{D}_1(q)}{\nu q^2} \epsilon_{\beta\eta\rho} q_\rho \epsilon_{\alpha\beta\gamma}(-i) q_\delta b_\rho(\mathbf{k},t=0) [\frac{1}{q^2(\nu+\mu)} + \frac{\exp(-2t\nu q^2)}{q^2(\nu-\mu)}]$$

(15)

giving $\alpha_D = \frac{2S_3}{3} \frac{1}{\nu(\nu+\mu)} \int_q 2\frac{\tilde{D}(q)}{(\nu+\mu)q^2}$ for large $t$. The suffix $D$ refers to *growth* or the *direct* term, as opposed to *feedback* which we discuss in the next Sec.III B. The growth term is proportional to $|k|$ and diffusive decay proportional to $k^2$. The angular brackets represent averaging over the noise and initial-condition ensembles.
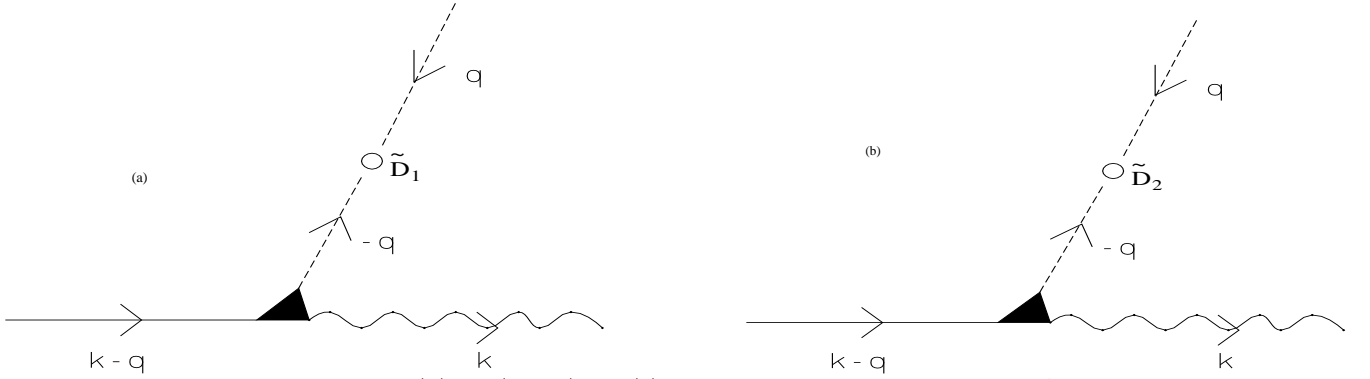


FIG. 1. Tree level diagrams for $< \mathbf{u}(\mathbf{q}) \times \mathbf{b}(\mathbf{k}-\mathbf{q}) >$. (a)Contribution to growth term $\alpha_D$: A solid line indicates a bare magnetic field response function, a broken line indicates a bare velocity response function, a 'o' joined by two broken lines indicates a bare velocity correlation function (proportional to $\tilde{D}_1$), a wavy line indicates a magnetic field, a solid triangle indicates a $ub$ vertex. (b) Contribution to feedback term $\alpha_F$: A solid line indicates a bare magnetic field response function, a broken line indicates a bare velocity response funtion, a 'o' joined by two broken lines indicates a bare magnetic field correlation function (proportional to $\tilde{D}_2$), a wavy line indicates a magnetic field, a solid triangle indicates a $ub$ vertex.
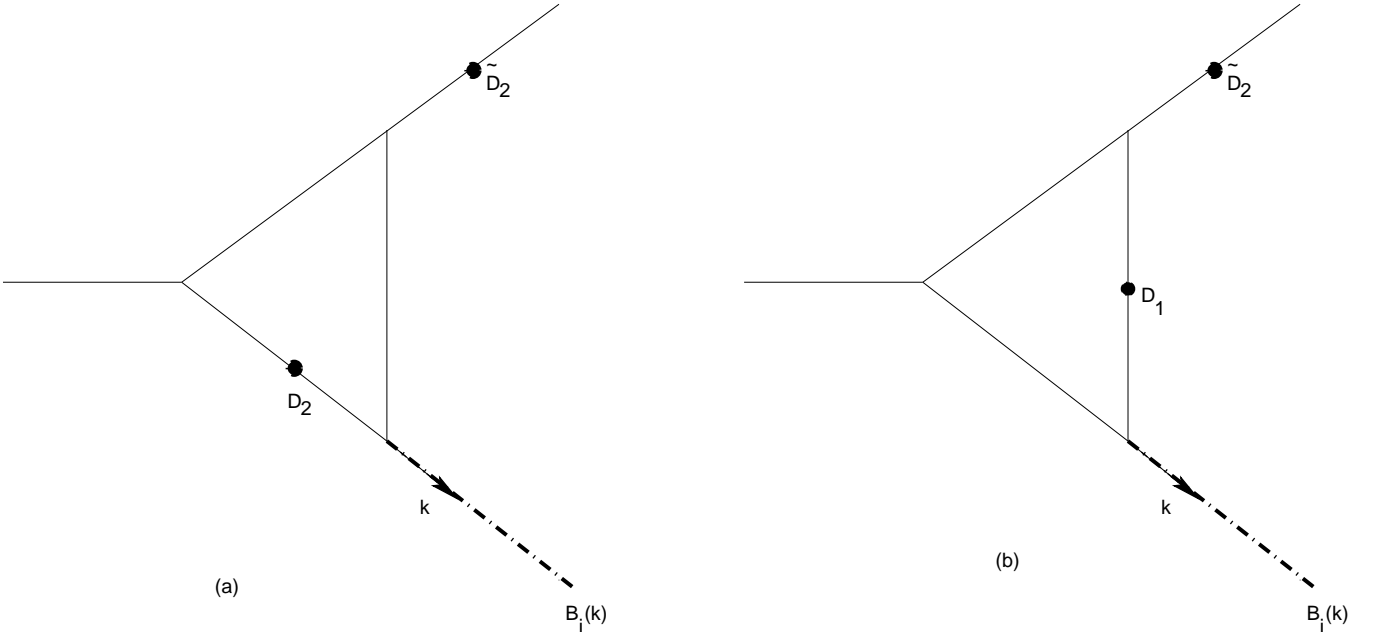


FIG. 2. Two one-loop diagrams contributing to $\alpha_F$. There are total six diagrams altogether.

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## B. Suppression of growth rate: Nonlinear feedback

When the magnetic fields become strong, it is no longer justified to neglect the feedback of the magnetic fields in the form of the Lorentz force. So we need to work with the *full* Eqs.(6) and (7). The ideas of OPE as elucidated in Sec.II are still valid for the full non-linear problem. But the value of $\alpha$ is expected to change from its value in the linear problem. In presence of the Lorentz force there is an additional contribution to $\alpha$ (Fig.1b). To evaluate that, we follow a diagrammatic perturbation approach similar to that described in the previous Section. Here also we restrict ourselves to the lowest order only (i.e., the tree level diagrams) though extension to higher orders is straight forward. We obtain

$$\langle (\mathbf{u} \times \mathbf{b}_i)_F \rangle = \langle \int_q \epsilon_{ijp} u_j(\mathbf{q}, t) b_p(\mathbf{k} - \mathbf{q}, t) \rangle \tag{16}$$

$$= \langle \frac{i}{2} \epsilon_{ijp} \int_q P_{jmn}(q) G_o^u(q, t - t_1) \quad b_m(\mathbf{q_1}, t_1) b_n(\mathbf{q} - \mathbf{q_1}, t_1) G_o^b(k - q, t, t = 0) b_p(\mathbf{k} - \mathbf{q}, t = 0) \rangle \tag{17}$$

which gives ($F$ refers to feedback)

$$\alpha_F B_i(\mathbf{k}, t) = i \epsilon_{ijp} \int_q P_{jmn}(q) e^{2\alpha_D |q| t - 2\mu q^2 t} b_n(\mathbf{k}, t) \frac{-2i \tilde{D}_2(q) \epsilon_{mps} q_s}{2\alpha_D |q| - 2\mu q^2}, \tag{18}$$

which, after some simplifications, yields,

$$\alpha_F(t) = \frac{2S_3}{3} \frac{4}{15} \int_q \frac{\tilde{D}_2(q, t) q^2}{\alpha_D |q| - 2\mu q^2}, \tag{19}$$

where $\tilde{D}_2(q, t) = \exp[2\alpha_D |q| t - 2\mu q^2 t] \tilde{D}_2(q)$ is a growing function of time for small wavenumbers. As before, angular brackets refer to averaging over noise and initial-condition ensembles. Thus $\alpha_F$ grows in time.

Since, at any finite time $t$, when the non-linear feedback on the velocity field due to the Lorentz force is nolonger negligible, both $\alpha_D$ and $\alpha_F$ are non-zero and we get

$$\alpha_D = -\frac{2S_3}{3} \int \frac{d^3q}{(2\pi)^3} \frac{\tilde{D}_1(q)}{\nu[|(\alpha_D + \alpha_F)q| - (\nu + \mu)q^2]},$$
$$\alpha_F = \frac{2S_3}{3} \frac{4}{15} \int \frac{d^3q}{(2\pi)^3} \frac{\tilde{D}_2(q, t) q^2}{|(\alpha_D + \alpha_F)q| - 2\mu q^2}, \tag{20}$$

with

$$\tilde{D}_2(q, t) = \exp[2(\alpha_D - \alpha_F(t))|q| t - 2\mu q^2 t] \tilde{D}_2(q). \tag{21}$$

Equations (20) and (21) are to be solved self-consistently [17]. Thus the net growth rate is proportional to $|(\alpha_D + \alpha_F)k|$ for the mode $B_i(\mathbf{k}, t)$. The expressions (20) have apparent divergences at finite $q$; so in perturbative calculations one should treat the $\alpha$-terms as perturbations which remove these divergences. This problem is akin to that in Kuramoto-Shivashinsky equation for flame front propagation [25]. So the expressions for $\alpha_D$ and $\alpha_F$ are

$$\alpha_D = \frac{2S_3}{3} \int \frac{d^3q}{(2\pi)^3} \frac{\tilde{D}_1(q)}{\nu(\nu + \mu)q^2}, \tag{22}$$

$$\alpha_F = -\frac{2S_3}{3} \frac{4}{15} \int \frac{d^3q}{(2\pi)^3} \frac{\tilde{D}_2(q, t)}{2\mu q^2}, \tag{23}$$

which do not have any finite wavevector singularity. Expressions (23) are obtained, as mentioned before, by truncating the perturbation series at the tree level. Extensions to higher orders are straight forward. Illustrative examples of higher order diagrams have been shown in Fig.2.

Let us now consider various $k$ dependences of $\tilde{D}_1(k)$ and $\tilde{D}_2(k)$. When the background velocity field is driven by the Navier-Stokes equation with a conserved noise (thermal noise) one requires that $D_1(k) = D_1 k^2$, $\tilde{D}_1 = \tilde{D}_1 |k|$, giving $\langle u_i(\mathbf{k}, t) u_i(-\mathbf{k}, t) \rangle = constant$. If we assume similar $k$-dependences for $\langle b_i(\mathbf{k}, 0) b_i(-\mathbf{k}, 0) \rangle$ then we require $D_2(k) \sim$ constant and $\tilde{D}_2(k) = \frac{\tilde{D}_2}{|k|}$. These choices yield

$$\alpha_D = \frac{2S_3}{3} \int \frac{d^3q}{(2\pi)^3} \frac{\tilde{D}_1|q|}{\nu(\nu+\mu)q^2},$$

$$\alpha_F = -\frac{2S_3}{3} \frac{4}{15} \int \frac{d^3q}{(2\pi)^3} \frac{\exp[2(\alpha_D - \alpha_F)|q|t]}{2\mu|q|}, \tag{24}$$

which remain finite even if the system size diverges.

A fully developed turbulent state, characterised by K41 energy spectra, is generated by $D_1(k) \sim k^{-3}$ and $\tilde{D}_1(k) = \tilde{D}_1 k^{-4}$. In addition if we assume that the initial magnetic fields correlation also have K41 scaling then $D_2(k) \sim k^{-5/3}$ and $\tilde{D}_2(k) = \tilde{D}_2 k^{-8/3}$. If one starts with a K41-type initial correlations for the magnetic fields, then at a later time the scale dependence for the magnetic field correlations are likely to remain same; only the amplitudes grow. Notice that the spectra diverge as wavevector $k \to 0$, i.e., as the system size diverges. This is a typical characteristic of fully developed turbulence. For such a system we find

$$\alpha_D = \frac{2S_3}{3} \int \frac{d^3q}{(2\pi)^3} \frac{\tilde{D}_1 q^{-4}}{\nu(\nu+\mu)q^2} = \frac{2S_3}{3} \int \frac{d^3q}{(2\pi)^3} \frac{2M^{-1}D_1}{\nu^2(1+P_m)q^6},$$

$$\alpha_F = -\frac{2S_3}{3} \frac{4}{15} \int \frac{d^3q}{(2\pi)^3} \frac{\tilde{D}_2(t)q^{-8/3}}{2\mu} = -\frac{2S_3}{3} \frac{4}{15} \int \frac{d^3q}{(2\pi)^3} \frac{\tilde{D}_2(t)q^{-8/3}}{2P_m\nu}. \tag{25}$$

The notable difference between the expressions Eqs.(24) and (25) for the $\alpha$ coefficients is that the $\alpha$ coefficients diverge with the system size if the energy spectra are singular in the infra red limit (as in for fully developed turbulence). These divergences are reminiscent of the divergences that appear in critical dynamics [18] which are handled by renormalisation group methods.

In general, at early times (small $\alpha_F$), $\alpha_F$ increases exponentially in time. The growth rate of $\alpha_F$ decreases with time. Since $\alpha_D$ and $\alpha_F$ have different signs, $|(\alpha_D + \alpha_F)| \to 0$ as time $t$ increases. Thus the net growth rate comes down to zero. Hence, Eq.(24) and Eq.(25) suggest that the early-time growth and late time saturation of magnetic fields take place for different types of background velocity correlations and initial magnetic field correlations. Therefore dynamo instability and its saturation are rather intrinsic properties of the 3$d$MHD equations with broken reflection invariance. One may also note that for $K41$-type of correlations (singular in the infrared limit) one has forward cascade of kinetic energy [24]: This is because energy is fed into the system mostly in the large scale (i.e., for small $k$) whereas, dissipation acts primarily in the small scales (large $k$), resulting into a cascade of energy from the large- to small- scales. On the other hand, for correlations smooth in the infra red limit, there is no such cascade. These results indicate that the existence of the dynamo mechanism does not require any special background velocity field spectrum, though the value of the $\alpha$-coefficient depends upon it. Our results also suggest that these processes may take place for varying magnetic Prandtl number $P_m = \mu/\nu$. The above analysis crucially depends on the fact that $\alpha_F$ and $\alpha_D$ have opposite signs, which, in turn, imply that $\tilde{D}_1$ and $\tilde{D}_2$ have same signs. We have already seen that in a physically realisable situation where parity is broken entirely due to the global rotation, $\tilde{D}_1$ and $\tilde{D}_2$ indeed have the same sign.

In the *first order smoothing approximation* [1,26] in the kinematic limit, to calculate $\langle \mathbf{u} \times \mathbf{b} \rangle$ one considers only the Induction equation as $\mathbf{u}$ is supposed to be given. However when one goes beyond the kinematic approximation, one has to consider the Navier-Stokes equation as well. Thus in the first-order smoothing approximation one writes the equations for the fluctuations $\mathbf{u}$ and $\mathbf{b}$ as (to the first order)

$$\frac{\partial \mathbf{b}}{\partial t} \approx \nabla \times (\mathbf{u} \times \overline{\mathbf{B}}) + \nabla \times (\overline{\mathbf{U}} \times \mathbf{b}), \tag{26}$$

and

$$\frac{\partial \mathbf{u}}{\partial t} \approx \dots + (\overline{\mathbf{B}}.\nabla)\mathbf{b}, \tag{27}$$

where the ellipsis refer to all other terms in the Navier-Stokes equation and $\overline{\mathbf{B}}$ and $\overline{\mathbf{U}}$ are the large scale (*mean field*) part of the velocity and magnetic fields [1,26]. With these we can write

$$\langle \mathbf{u} \times \mathbf{b} \rangle_i = \langle \epsilon_{ijp} u_j b_p \rangle = \langle \epsilon_{ijp} u_j B_m \frac{\partial}{\partial x_m} u_p \rangle + \langle \epsilon_{ijp} b_p B_m \frac{\partial}{\partial x_m} b_j \rangle \equiv \alpha_{im} B_m + \dots \tag{28}$$

Here the ellipsis refer to non-$\alpha$ terms in the expansion of $\langle \mathbf{u} \times \mathbf{b} \rangle$ (see Eq.(4)). Thus for isotropic situations $\alpha = \frac{\tau}{3}[-\langle \mathbf{u}.(\nabla \times \mathbf{u}) \rangle + \langle \mathbf{b}.(\nabla \times \mathbf{b}) \rangle]$ where $\tau$ is a correlation time. Thus $\alpha$ is proportional to the difference in the fluid

and magnetic *torsalities* [4], (fluid helicity being the same as fluid torsality and magnetic helicity being proportional to magnetic torsality) a result obtained in [4,7] using other methods and approximations. Note that Eqs. (21) and (23) are very similar to but not exactly the one that were obtained in [7] (in our notations $\tilde{D}_1$ is proportional to fluid torsality (or fluid helicity) and $\tilde{D}_2$ is proportional to magnetic torsality). We ascribe this difference to the essential difference between a two-scale approach and our dirgrammatic perturbation theory which, we believe is more suitable for handling continuous kinetic and magnetic spectra.

## IV. HYDRODYNAMIC LIMIT OF DYNAMO GROWTH

We have seen that in Eqs.(25) the $\alpha$-coefficients diverge in the hydrodynamic ($k \rightarrow 0$) limit which calls for a renormalisation group (RG) analysis as a natural extension of our diagrammatic perturbative calculations. In fully developed 3$d$MHD, in the steady state, correlation and response functions exhibit dynamical scaling with the *dynamic exponent $z = 2/3$* [27,28] (for a different approach see [29]), which means renormalised viscosities (kinetic as well as magnetic) diverge $\sim k^{-4/3}$ for a wavenumber $k$ belonging to the inertial range. Even for decaying MHD with initial K41-type correlations this turns out to be true [30] where equal time correlations exhibit dynamical scaling with $z = 2/3$. The question is, what it is in the initial transient of dynamo growth ($t \ll$ saturation time). We examine this in a renormalization group framework. Since we are interested in the early growth, we neglect the Lorentz force and work with Eq.(7) inconjunction with the initial magnetic field correlations and noise correlation given by Eq.(11). As before, we assume a statistical steady state for the velocity field. It is well-known that correlations $\langle u_i(\mathbf{k},t)u_j(-\mathbf{k},0)\rangle$ exhibit scaling form $k^{-d-2\chi}h(tk^z)$ where $\chi$ and $z$ are the spatial scaling and dynamical exponents respectively [24] where $h$ is a scaling function. The Galilean invariance of the MHD equations constraints these exponents to obey the relation $\chi + z = 1$ [24,28,33]. In addition to that, for fully developed turbulence due to non-renormalization of the noise-correlators [cf. Eq.(8)] the exponents are fully determined: $z = 2/3$, $\chi = 1/3$, which means the renormalised fluid viscosity diverges as $k^{-4/3}$ in the limit wavevector $k \rightarrow 0$. During early growth, equal-time magnetic field correlations $\langle b_i(\mathbf{k},t)b_j(-\mathbf{k},t)\rangle$ are expected to exhibit a scaling form $k^{-d-2\chi_b}m(tk_b^z)$ ($t \ll$ saturation time) where $\chi_b$ and $z_b$ are the magnetic spatial scaling and dynamical exponents respectively, and $m$ is a scaling function. Similar conditions arising from the Galilean invariance and non-renormalization of the initial K41-like magnetic field spectrum determines $z = z_b = 2/3$ and $\chi = \chi_b = 1/3$. We perform a renormalization group analysis following [19,24,30]. As mentioned earlier, the $\alpha$-term is treated as a perturbation. In a renormalisation-group transformation, one integrates out a shell of modes $\Lambda e^{-l} < q < \Lambda$, and and simultaneously rescales length scales, time intervals and fields through $\mathbf{x} \rightarrow e^l \mathbf{x}$, $t \rightarrow e^{lz}t$, $\mathbf{u} \rightarrow e^{l\chi}\mathbf{u}$, $\mathbf{b} \rightarrow e^{l\chi}\mathbf{b}$. This has the effect that the nonlinearities are affected only by naive rescaling (this, a consequence of the Galilean invariance of the 3$d$MHD equations, essentially implies that the diagrammatic corrections to the nonlinearties vanish in the long wavelength limit). The variances Eq.(8), which diverge at low wavenumbers remain unrenormalised and thus affected only by rescaling. There are however fluctuations corrections to $\mu$ and $\alpha_D$ which we evaluate at the lowest order. The resulting RG flow equations for $\mu$ and $\alpha_D$, obtained in a one-loop calculation are

$$\frac{d\mu}{dl} = \mu[z_b - 2 + A_1 \frac{D_1}{\nu^2(\nu + \mu)\Lambda^4}], \tag{29}$$

$$\frac{d\alpha_D}{dl} = \alpha_D[z_b - 1 + A_2 \frac{\tilde{D}_1}{\alpha_D\nu(\nu + \mu)\Lambda^3}], \tag{30}$$

where $A_1$, $A_2$ are numerical constants. Equations (30) and (30) are similar to those presented in Ref. [32] [Eqs. (10.13) and (10.14)] but not exactly same. The differences arise mainly (apart from some detail technical differences in the perturbation theories involed) from the fact that in Ref. [32] the expressions for the $\alpha$-coefficients were derived for a given variance of the velocity field. In contrast, we use the Navier-Stokes equation, driven by a stochastic force of given variance, in place of a given velocity variance. By substituting the value of the exponents in Eqs.(30) and (30) we find *renormalized* (i.e., wavevector dependent) $\alpha_D(k) \sim \alpha_D k^{-1/3}$, $\mu(k) \sim k^{-4/3}$ in the hydrodynamic ($k \rightarrow 0$) limit. Thus in that limit, the effective dynamo equation takes the form

$$\frac{\partial b_i}{\partial t} = (\alpha_D - \mu)k^{2/3}b_i + .... \tag{31}$$

where the ellipsis refer to non-linear terms and $i$ refers to the growing mode. Thus, in the hydrodynamic limit, there is growth of the magnetic fields only if $\alpha_D - \mu > 0$. This can happen only if the renormalised magnetic viscosity is less than a critical value, set by $\alpha_D$, i.e., the kinetic helicity. In terms of the Ekman number $M$ this condition

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means $M \stackrel{>}{\sim} 1/2$ for anti dynamo, i.e., no growth, equivalently $M \lesssim 1/2$ for growth of the magnetic fields. This can be achieved in two ways, namely by increasing rotation, keeping the magnetic viscosity (or the magnetic Reynolds number) constant, or decreasing the magnetic viscosity (i.e., increasing the magentic Reynolds number) for a constant rotation. This conclusions are in good agreement with the numerical results of Ref. [14]. Since renormalised magnetic viscosity increases with its bare (microscopic) value, it suggests that bare magnetic viscosity must be less than a critical value for growth to be possible. Thus our RG results qualitatively explain the numerical results of Kida *et al* [31] who they found that unless magnetic hyperviscosity was less than a critical value there was no growth (it can be easily argued that a hypermagnetic viscosity gives rise to a magnetic viscosity in the longer scale and hence their result in effect imposes a critical value of the magnetic viscosity). In our model $\alpha$-effect is proportional to $\tilde{D}$ which in turn is proportional to the global rotation frequency. Hence our results suggest that $\alpha$-effect is likely to grow with increasing rotational speed which is in agreement with the results of Ref. [9]. On the other hand, if the background velocity and the initial magnetic field correlators do not have an infra red singularity (i.e., when the correlators $\sim k^2$) there is no fluctuation correction to the magnetic viscosity and to the $\alpha$-coefficient resulting in the fact that the growth term ($\propto k$) dominates over the dissipation ($\propto k^2$) for sufficiently small wavenumber $k$, leading to growth even for arbitrarily large magnetic viscosity. Therefore, there is no critical $R_m$. Thus the effects of the infrared divergences that appear in the expressions for the $\alpha$-coefficients [Eq. (25)] are quite significant: They indicate, as for the driven diffusive nonequilibrium systems with diverging kinetic coefficients in the hydrodynamic limit [19,24], divergence of time-scales in the hydrodynamic limit. Since, the $\alpha$-term in Eq. (7) is proportional to wavenumber $k$, the time-scale of growth of the mode with wavenumber $k$ is $O(\alpha k)$. This remains true, even in the hydrodynamic limit, for the case when there is no divergence in the $\alpha$-coefficients. In contrast, when the $\alpha$-coefficient diverge in the infra red limit, the growth rate changes *qualitatively* from its linear dependence on wavenumber $k$ in the hydrodynamic limit. For example, with the the background velocity correlations and the initial magnetic field correlations given by Eq. (8), the $\alpha$ coefficients diverge as $k^{-1/3}$ in the long wavelength limit. Hence, the effective growth rate is changed to $\alpha(k)k \sim k^{2/3}$. A full self-consistent calculation (when feedback due to the Lorentz force cannot be neglected) for the $\alpha$-coefficients require simultaneous solutions of the self-consistent expressions for magnetic Prandtl number, magnetic-to kinetic- energy ratio and the $\alpha$-coefficients which can be handled in our scheme of calculations. The self-consistent solutions are expected to be influenced by the degree of crosscorrelations between the velocity and magnetic fields [33].

So far, we have assumed that both $D_1(k)$ and $\tilde{D}_1(k)$ have the same infra red singularity ($D_1(k) \sim k^{-5/3}$ and $\tilde{D}_1(k) \sim k^{-5/3}$). This need not be the case always. However, if $\tilde{D}_1(k)$ is non-singular then $\alpha_D$ does not diverge. As a result, the growth rate is just $\alpha_D k$ even in the hydrodynamic (long wavelength) limit. Effective dissipation, however, will still be $\sim k^{2/3}$ and thus it will dominate over $O(k)$ growth. Therefore, there will be no growth in the hydrodynamic limit. Thus, our analyses suggest that in any fully developed turbulent system with $\alpha$-effect, helicity spectrum (given by $\tilde{D}_1(k)$) should be as singular as the kinetic energy spectrum (given by $D_1(k)$).

## V. CONCLUSIONS

In conclusions, we have calculated expressions for the $\alpha$-coefficients in a diagrammatic perturbation theory on a minimal model for arbitrary background velocity and initial magnetic field correlations, and fluid and magnetic viscosities. We show that the parity breaking parts of the velocity and magnetic field variances must have the same sign, which is the case in any physical system. We explicitly show that the processes of early growth and late-time saturations may take place independent of any special velocity and initial magnetic field correlations. Even though our explicit calculations were done by using simple initial conditions for the calculational convenience, the results that we obtain are general enough and it is apparent that the feedback mechanism is qualitatively independent of the details of the initial conditions and force correlations. one may note that for one of the force/initial correlations there is no kinetic energy cascade in the conventional sense but we still find dynamo action. It is quite reasonable to expect that our results should be valid for more realistic initial conditions also. In effect we have explicitly demonstrated the robustness and generality of the dynamo mechanism and that the dynamo mechanism is an intrinsic property of the $3d$MHD equations. We have also shown, within our RG analysis, that the magnetic viscosity should be less than a critical value for growth of magnetic fields a result which was previouly observed in numerical simulations. We conclude the existence of a critical Ekman number for K41-type correlations: We find growth only when $M \lesssim 1/2$, confirming recent numerical results. This is easily understood in our framework. The issue of divergent effective viscosities in the inertial range assumes importance as it may help to overcome some of the non-linear restrictions as discussed by Vainshtein and Cattaneo [6]. A system of magnetohydrodynamic turbulence in a rotating frame, after the saturation time (i.e., after which there is no net growth of the magnetic fields) belongs to the universality class

of usual three-dimensional magnetohydrodynamic turbulence in a laboratory. This can be seen easily in both the lab and the rotating frames; the critical exponents characterising the correlation functions can be calculated exactly by using the Galilean invariance and noise-nonrenormalisation conditions [24,28]. An important question, which remains open for further investigations, is the multiscaling properties of the velocity and the magnetic field structure functions at various stages of the growth of the magnetic fields. In what concerns an experimental observation of our results, one should add that even though it is not easy to verify our results in an experimental set up, numerical simulations of Eqs.(6) and (7) with the variances (11) with different $k$-dependences can be performed to check these results.

## VI. ACKNOWLEDGEMENT

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